How to make the gravitational action on noncompact space finite

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The recently proposed technique to regularize the divergences of the gravitational action on noncompact space by adding boundary counterterms is studied. We propose a prescription for constructing boundary counterterms which are polynomial in the boundary curvature. This prescription is efficient for both asymptotically anti−de Sitter and asymptotically flat spaces. Being mostly interested in the asymptotically flat case we demonstrate how our procedure works for known examples of noncompact spaces: Eguchi-Hanson metric, Kerr-Newman metric, Taub-NUT, Taub-bolt metrics, and others. Analyzing the regularization procedure when the boundary is not a round sphere we observe that our counterterm helps to cancel the large r divergence of the action in the zero and first orders in small deviations of the geometry of the boundary from that of the round sphere. In order to cancel the divergence in the second order in deviations a new quadratic in boundary curvature counterterm is introduced. We argue that the cancellation of the divergence for finite deviations possibly requires an infinite series of (higher order in the boundary curvature) boundary counterterms.

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I. INTRODUCTION

The classical dynamics of gravitational field (metric $g_{\mu\nu}$ on d-dimensional manifold M^d) is determined by the Einstein-Hilbert (EH) action

$$W_{EH}[g]=-\frac{1}{16\pi G}\left(\int_{M^d}(R+2\Lambda)+2\int_{\partial M^d}K\right), \quad (1.1)$$

where the boundary term proportional to the extrinsic curvature K of the boundary ∂M should be added in order to make the variational procedure of the action (when only metric but not its normal derivative is fixed on the boundary) well defined [1,2]. When the manifold M is noncompact one considers a sequence of compact manifolds M_r with the boundary ∂M_r parametrized by the radius r such that $M_r \to M$ for large r. The functional (1.1) on a noncompact manifold M then should be understood as a result of the limit $W_{EH}[M_r \to M]$. It is, however, a well-known problem that this limiting procedure is not well defined since $W_{EH}[M_r]$ diverges in the limit of large r. Therefore, the limiting procedure should be accompanied by some regularization. The traditional way [3] of handling this problem is to subtract a contribution of some reference metric g_0 that matches suitably the asymptotic and topological properties of the metric g. The choice of the metric g_0 is interpreted as fixing the vacuum state. However, such a reference metric does not always exist which makes this subtraction procedure quite uncertain.

It was realized recently that when the space M is asymptotically AdS (rather than asymptotically flat) one can take an alternative route. In the context of the AdS-conformal field theory (CFT) correspondence a general analysis (based on previous mathematical works [4,5]) of the divergences of the EH action for AdS space was done in [6]. Inspired by the AdS-CFT correspondence, Balasubramanian and Kraus [7] have proposed to add to the action (1.1) a counterterm which is a functional of the curvature invariants of the induced metric h_{ij} on ∂M_r. The role of this term (which does not affect the gravitational equations in the bulk) is to cancel appropriately the large r divergence appearing in $W_{EH}[M_r]$. The counterterm $W_{ct}[h_{ij}]$ can be arranged as an expansion in powers of the curvature of the boundary metric. The first few terms are the following [6–8]:

$$W_{ct}^{bk}=\frac{1}{16\pi G}\int_{\partial M_r^d}\sqrt{h}\left[\frac{2(d-2)}{l}+\frac{l}{d-3}\mathcal{R}+\frac{l^3}{(d-5)(d-3)^2}\left(\mathcal{R}_{ij}^2-\frac{(d-1)}{4(d-2)}\mathcal{R}^2\right)+\cdots\right], \quad (1.2)$$

where $\mathcal{R}_{ij}$ and $\mathcal{R}$ are, respectively, the Ricci tensor and Ricci scalar of the boundary metric, and l is the AdS radius related to the cosmological constant as $\Lambda=(d-1)(d-2)/2l^2$. The terms (1.2) are sufficient to cancel divergences for $d\leqslant 7$. On the other hand, the leading divergence in any d is always removed by the term (first introduced in Eq. [9]) in Eq. (1.2) which is proportional to the area of the boundary.[1] It should be mentioned that introducing counterterms which are polynomial in the boundary curvature one is able to cancel all divergences of the action (1.1) but not the logarithmic one [appearing when $(d-1)$ is even]. The later divergence can be canceled by adding a counterterm which is not polynomial in curvature. For example, for $d=3$ it is the term $\mathcal{R}\ln\mathcal{R}$ that should be added. In higher dimensions there is ambiguity in choosing such terms. Up to this subtlety the procedure of introducing the counterterms (1.2) is universal and well defined.

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[1]The extrinsic curvature of the asymptotic boundary of AdS space is constant, $K=(d-1)/l$. Therefore, the first term in Eq. (1.2) can be presented as a surface integral of K. For $d=3$ this was observed in [10]

Encouraged by this example one could try to construct an appropriate boundary term which cancels the leading divergence for asymptotically flat space. This term can be found but it is not an analytic function of the boundary curvature [11,12]

$$W_{ct}^{LM} = -\frac{c_{LM}}{16\pi G}\int_{\partial M_r}\sqrt{\mathcal{R}}. \tag{1.3}$$

The constant c_{LM} depends on the topological type of boundary at large r. For the Schwarzschild like metric (when boundary is topologically $S_1\times S_{d-2}$) one has $c_{LM}=-2\sqrt{(d-2)/(d-3)}$. Not requiring the counterterm to be an analytic function of the curvature one can also construct a term interpolating between expressions (1.2) and (1.3) [12,13]:

$$W_{ct}^{int} = \frac{1}{16\pi G}\ \frac{2(d-2)}{l}\int_{\partial M_r}\sqrt{1+\frac{l^2}{(d-3)(d-2)}\mathcal{R}+\cdots}. \tag{1.4}$$

Indeed, for large r the boundary curvature $\mathcal{R}$ vanishes and we need to take the limit of small $\mathcal{R}$ in Eq. (1.4) in order to get Eq. (1.2). On the other hand, the asymptotically flat case is obtained by taking the limit of large l in Eq. (1.4). The expression (1.3) then is reproduced. We stress that this interpolation exists only for the choice of the constant c_{LM} in Eq. (1.3) as in the case of the Schwarzschild black hole. The boundary then is $S_1\times S_{d-2}$. For other types of the boundary the expression (1.4) does not match Eq. (1.3) in the limit of large l.

There are, however, reasons to think that it is not an option to drop the analyticity in the proposed procedure of introducing the boundary counterterms. The form of the counterterms then is not unique and, in fact, quite ambiguous. Indeed, for asymptotically flat space, not only $\sqrt{\mathcal{R}}$ but any function $(\mathcal{R}_{ij}^2)^{1/4}$, $(\mathcal{R}_{ijkl}^2)^{1/4}$ or even higher roots of higher power curvature invariants can be chosen as a candidate for the counterterm. In the asymptotically AdS case we also can take as a counterterm any function $f(l^2\mathcal{R})$ that approaches $(1+[l^2/2(d-3)(d-2)]\mathcal{R})$ for small $\mathcal{R}$. Among these functions, in particular, there are ones which do not have the well-defined flat space ($l\to\infty$) limit.

Another reason why it is not desirable to use nonanalytic boundary counterterms appears from the consideration of the EH term in quantum theory. Any quantum field makes a contribution to the EH action. In fact, this contribution is UV divergent and we have to renormalize the Newton's constant G (and cosmological term Λ) in order to handle these divergences. The natural question is then whether the structure of the classical action $W_{EH}+W_{ct}$ is preserved under quantum corrections and whether it remains the same after renormalization. For the EH action (1.1), this question was addressed in [14]. It was found that the exact balance between the bulk and boundary parts in the quantum action is the same as in the classical action (1.1). Hence the renormalization of only Newton's constant (the Λ term was dropped in [14] but this does not affect the main conclusion) is sufficient to regularize both the bulk and boundary UV divergences. In fact this statement is quite obvious in the case of matter minimally coupled to gravity. One just has to impose Dirichlet or Neumann conditions on the quantum field on the boundary ∂M. In the nonminimal case the boundary condition should be chosen of a mixed type in order to make this statement valid. Analyzing now this problem for the action $W_{EH}+W_{ct}$ with the counterterm in the form (1.3) or (1.4) it is hard to imagine how this structure can be preserved in the quantum case since only terms analytic in the boundary curvature are known to appear in the quantum effective action (at least in its UV divergent part) on a manifold with a boundary.

Concluding our brief analysis we see that the nonanalytic boundary counterterms are likely not allowed in an unambiguous and universal procedure of the regularization of the gravitational action. The purpose of this paper is to propose another way of constructing the counterterms remaining in the class of the functions which are analytic in curvature. In the asymptotically flat case, we are not going to generalize the AdS prescription (1.2) in the part of the dependence of the boundary counterterm on the boundary metric. Instead, keeping the general structure of the counterterm as in Eq. (1.2), we define a scale parameter l^* (analogous to the parameter l in the AdS case) which characterizes the global geometry of the space (in fact, it is the coordinate invariant *diameter* of the space) and can be used in the constructing the counterterms in the same fashion as in the AdS case. The prescription, thus, works universally both in asymptotically AdS and asymptotically flat cases and deals only with the analytic structure of the counterterms.

II. PROPOSAL

It should be noted that the counterterm (1.2) is not an off-shell quantity. In fact, it contains some information about the asymptotic bulk geometry. Namely, the space-time is supposed to be anti–de Sitter space with radius l. The role of the parameter l in AdS space is twofold. First, it measures the curvature of the bulk geometry [Ricci scalar $R=-d(d-1)/l^2$]. Second, it measures the size of the space: l is that quantity which relates the volume of AdS space, $V(M_r)$, and area of its boundary, $A(\partial M_r)$, in the limit of large r, i.e., $l\sim V(M_r)/A(\partial M_r)$. This relation is the key one [15] in the holographic correspondence between the gravitational theory in the bulk of AdS space and conformal field theory on the boundary. As we said above, our idea is to introduce the parameter l^* which for asymptotically flat space plays a role similar to that of the parameter l in the AdS case. Since it is not possible in general to find any scale parameter universally related to the curvature if the space is asymptotically flat, it is the holographic relation which we are going to generalize. Note that in our prescription we do not require the metric to satisfy any equations of motion and in this sense it is an off-shell prescription. We only demand that (in the case of zero cosmological constant) the curvature of the space-time die sufficiently fast with large r so that the bulk integral $\int_{M_r}R$ converges in the large r limit. The only divergence of the gravitational action (1.1) then comes from the boundary term $2\int_{\partial M_r}K$. Note also that we will be mostly considering the leading divergence of the action.

Consider the compact manifold M_r with boundary ∂M_r parametrized by ''radial'' coordinate r in an appropriate coordinate system. Let $V(M_r)$ be the invariant volume of M_r and $A(\partial M_r)$ be the area of the boundary ∂M_r. Define the *diameter* l^* of the manifold M_r as follows:

$$l^* = \frac{V(M_r)}{A(\partial M_r)}. \tag{2.1}$$

Consider now a sequence of compact manifolds M_r approaching the noncompact manifold M in the limit of large r. The diameter l^* then, in general, becomes a function of r. Defying the gravitational action $W_{gr}[M]$ as the limit of the actions $W_{gr}[M_r]$ for large r we want it to be finite as $r \to \infty$. The action we propose takes the form

$$W_{gr}[M_r] = W_{EH}[M_r] + W_{ct}[\partial M_r], \tag{2.2}$$

where, as in the AdS case (1.2), the boundary counterterm

$$W_{ct}[\partial M_r] = -\frac{1}{16\pi G}\frac{c(\gamma)}{l_r^*}\int_{\partial M_r}\sqrt{h} \tag{2.3}$$

is proportional to the area of the boundary.

First, we want to demonstrate that by adding the counterterm (2.3) we do not change the Einstein equations following from the EH action. We fix finite r and consider small variations of the metric in the bulk assuming the induced metric on the boundary ∂M_r is fixed. The diameter (2.1) changes under variation of the metric in the bulk. At first sight it seems that this may result in rather complicated equations when the action (2.2) is varied. However, it is quite surprising that the presence of the extra term (2.3) has the same effect in the field equations as that of the effective cosmological term $\Lambda_{eff} = -c(\gamma)/2(l^*)^2$:

$$\delta W_{gr} = -\frac{1}{16\pi G}\int_{M_r}\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda_{eff}\ g_{\mu\nu}\right). \tag{2.4}$$

So for the boundary placed at finite r the extra boundary term (2.3) shows up in the gravitational equations in the form of the finite cosmological term Λ_{eff}. Considering the sequence of boundaries parametrized by increasing r we find a sequence of bulk metrics described by Einstein equations with decreasing (since l^* is growing as r for asymptotically flat space) cosmological constant. In the limit of infinite r the quantity Λ_{eff} disappears and the gravitational equations remain unaffected.

In order to show that the gravitational action (2.2), (2.3) is indeed less divergent than Eq. (1.1) and determine the coefficient $c(\gamma)$ let us consider on M^d the coordinate system $(\chi, x^i, i=1,\dots,d-1)$ where the metric looks as

$$ds^2 = d\chi^2 + h_{ij}(\chi,x)dx^i dx^j. \tag{2.5}$$

The compact manifold M_r is determined by the range of the radial coordinate, $0 \leqslant \chi \leqslant r$. The boundary ∂M_r stays at $\chi = r$ and $h_{ij}(r,x)$ is the metric induced on the boundary.

The area of the boundary ∂M_r and the volume of M_r are given by

$$A(\partial M_r) \equiv A(r) = \int \sqrt{h(x,r)}\,d^{d-1}x,$$

$$V(M_r) \equiv V(r) = \int^r A(\chi)\,d\chi. \tag{2.6}$$

Assume that for large r the area function $A(r)$ is represented by the series

$$A(r) = A_0 r^\gamma + A_1 r^{\gamma-1} + B r^{\gamma-1}\ln r + \cdots, \tag{2.7}$$

where A_0, A_1, B are some coefficients and the ellipsis stands for the subleading terms. Then for the volume of M_r we have that

$$V(r) = \frac{A_0}{\gamma+1}r^{\gamma+1} + \left(\frac{A_1}{\gamma} - \frac{B}{\gamma^2}\right)r^\gamma + \frac{B}{\gamma}\ln r + \cdots. \tag{2.8}$$

The parameter $\gamma > 0$ is the coordinate invariant; it shows how the area of ∂M_r (or volume of M_r) grows for large r. The radius l^* defined by the relation (2.1) then reads

$$l^* = \frac{r}{\gamma+1} + \left(\frac{1}{\gamma(\gamma+1)}\frac{A_1}{A_0} - \frac{1}{\gamma^2}\frac{B}{A_0}\right) + \frac{1}{\gamma(\gamma+1)}\frac{B}{A_0}\ln r + \cdots. \tag{2.9}$$

For the extrinsic curvature of the boundary we have $\int_{\partial M_r} K = \partial_r A(r)$, so that the leading divergence of the EH action for large r is proportional to $r^{\gamma-1}$. Assuming that the bulk part of Eq. (2.2) converges for large r [this restricts the metric $h_{ij}(r,x)$ to grow asymptotically not faster than r^2] we find that the boundary part of Eq. (2.2) is given by

$$W_{gr}^{boundary} = -\frac{1}{16\pi G}\left(2\partial_r A(r) + \frac{c(\gamma)}{l^*(r)}A(r)\right). \tag{2.10}$$

Taking now the limit of infinite r we find that the leading divergence of the gravitational action cancels provided we choose the constant $c(\gamma)$ to be

$$c(\gamma) = -\frac{2\gamma}{\gamma+1}, \tag{2.11}$$

so that the regularized action

$$W_{gr} = \frac{1}{8\pi G}r^{\gamma-2}\frac{1}{\gamma}B + O(r^{\gamma-3}) \tag{2.12}$$

is finite if $\gamma \leqslant 2$. In some cases the logarithmic term in the expansion (2.7) is absent. Then the leading term in the action (2.12) is of the order $r^{\gamma-3}$. Thus, the adding the counterterm (2.3) guarantees the cancellation of the leading divergence. In order to remove the divergences still present in the action one has to introduce extra counterterms such as the term $l^*\int_{\partial M}\mathcal{R}$ or $(l^*)^3\int_{\partial M}\mathcal{R}^2$. We consider such terms in Secs. III and IV.

In order to determine l^* we have to have information about the whole manifold M. However, for the cancellation of the divergences in the gravitational action only the asymptotic behavior of l^* is important. Therefore, it would be desirable to define another quantity l^*_a as the asymptotic value of l^* for large r. It can be used (instead of l^*) in constructing the boundary counterterm (2.3). The advantage of using l^*_a is that the counterterm (2.3) then depends only on the asymptotic properties of the bulk metric and is not sensitive to what happens inside the manifold. The quantity l^*_a is not, however, uniquely defined since it depends on how many terms [as in Eq. (2.9)] we want to keep in the large r expansion of l^*. On the other hand, the freedom in choosing the coordinates in the asymptotic metric (2.5) also may result in an ambiguity in the definition of l^*_a. In all cases these ambiguities affect only the subleading terms in l^*_a and, eventually, in the gravitational action. Note in this context that picking up the first three terms in the expansion (2.9) and using this in the counterterm (2.3) we get exactly the same result (2.12) for the the leading part of the gravitational action as when l^* is used. Of course, we can add more subleading terms not changing this conclusion.

We still need to find an unambiguous and coordinate invariant notion of the asymptotic value of l^*. In order to get an idea of such a notion consider the large r (i.e., valid outside of some compact, large enough region of manifold M) expansion of the diameter l^* done in any appropriate coordinate system. The r-dependent terms in this expansion—for instance, the functions $\{r,\ln r,1,1/r,1/r^2,\dots\}$—can be considered as forming a basis in functional space and the large r expansion of l^* is just a decomposition of l^* along this basis. Among the elements constituting the basis there are ones which grow infinitely with r. In the above example only the functions r and $\ln r$ are such elements. Then, the projection of l^* onto the subspace spanned by the asymptotically growing elements is what we will call the leading asymptotic value l^*_a. For the expansion (2.9) we have

$$l^*_a=\frac{r}{\gamma+1}+\frac{1}{\gamma(\gamma+1)}\frac{B}{A_0}\ln r.$$

Note that by definition the constant term is not included in l^*_a. The quantity l^*_a appears to be unambiguous and coordinate invariant.

The gravitational action regularized by the counterterm (2.3) with l^*_a then reads

$$W^a_{gr}=\frac{1}{8\pi G}(A_1-B)r^{\gamma-2}+O(r^{\gamma-3}). \tag{2.13}$$

For $\gamma=2$ it takes the finite value which is different from Eq. (2.12). In many examples of four-dimensional metrics we consider below, the parameters B and A_1 are related as $B=2A_1$. Then the limit of large r in the expressions (2.12) and (2.13) for $\gamma=2$ gives rise to results opposite in sign, $W_{gr}=-W^a_{gr}$. It happens that—namely, for l^*_a—our regularization procedure gives the same result as the standard subtraction method. In all examples we present below the corresponding regularized action is a non-negative quantity. As a simple illustration of our regularization procedure consider the $d=4$ Schwarzschild metric

$$ds^2=g(\rho)d\tau^2+\frac{d\rho^2}{g(\rho)}+\rho^2ds^2_{S_2},\quad g(\rho)=1-\frac{2m}{\rho}, \tag{2.14}$$

where $0\leqslant\tau\leqslant 2\pi\beta_H$, $\beta_H=4m$. It takes the form (2.5) after performing the coordinate transformation $\chi=\int^\rho d\rho/\sqrt{g(\rho)}$. Asymptotically, we have $\chi=\rho-m\ln\rho$. The compact space M_r is defined as $0\leqslant\chi\leqslant r$. The area of the boundary ∂M_r behaves as

$$A(r)=2\pi\beta_H\Sigma_2(r^2-mr-2mr\ln r+\cdots) \tag{2.15}$$

for large r. In Eq. (2.15) we recognize the expansion (2.7) with $\gamma=2$. The diameter l^* defined above and its leading asymptotic values are

$$l^*=\frac{r}{3}-\frac{m}{3}\ln r+\frac{m}{3}+\cdots,$$

$$l^*_a=\frac{r}{3}-\frac{m}{3}\ln r. \tag{2.16}$$

Respectively, we have, for the regularized action (2.12) and (2.13),

$$W_{gr}=-\frac{4\pi m^2}{G},\quad W^a_{gr}=\frac{4\pi m^2}{G}. \tag{2.17}$$

In W^a_{gr} we recognize the standard expression for the action of the Schwarzschild metric [2,18,12].

The parameter γ in Eqs. (2.7)–(2.9) is an important characteristics of the asymptotic geometry of the manifold M^d. Demanding the bulk metric to approach asymptotically the (locally) flat metric, γ is restricted by topology of the asymptotic boundary ∂M. In the simplest case, when the size of ∂M^d grows equally in all directions [for example, if ∂M^d is a $(d-1)$-sphere and the asymptotic metric on M^d is $ds^2=dr^2+r^2ds^2_{S_{d-1}}$], we have $\gamma=d-1$. However, γ can be less than $(d-1)$ if, for example, asymptotic metric is $ds^2=\Sigma^n_{i=1}dz_i^2+dr^2+r^2ds^2_{S_{d-n-1}}$, where each coordinate z_i is compact. Then $\gamma=d-n-1$. It seems to follow from these examples that γ is related to the dimension of the spheric component in the boundary ∂M. In $d=4$ the locally flat metric may take the form $ds^2=dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)+(d\tau+2n\cos\theta d\phi)^2$. The surface of constant r is then the Hopf fiber bundle $S_3\rightarrow S_2$ with fiber S_1. Locally it looks as a direct product $S_1\times S_2$. However, the appropriate identifications (in τ and ϕ) and overlapping coordinate patches give the surface of constant r the topology of S_3. In this case $\gamma=2$ is the same as for the boundary $S_1\times S_2$.

In the asymptotically AdS case, the expansion (2.7) is not valid since the area

$$A(r)=e^{(d-1)r/l}l^{d-1}A_0[1+O(e^{-r/l})]$$

grows exponentially with r. As a result, the quantity l^* asymptotically takes the constant value

$$l^*=\frac{l}{d-1}+O(e^{-r/l}). \tag{2.18}$$

Therefore, the notion of the leading asymptotic value l_a^* defined above is not good for asymptotically AdS spaces. In this case, we have to define it as the first, constant, term in the large r expansion, so that we have $l_a^*=l/(d-1)$. Using this quantity in the counterterm (2.3) we reproduce correctly the first term in the AdS expression (1.2) provided the value $\gamma=d-2$ is used to define the constant $c(\gamma)$, Eq. (2.11). Note that in both the asymptotically AdS and asymptotically flat cases our prescription for l_a^* is to take the leading part in the large r expansion for l^*.

It is interesting to note that using the quantity l^* in Eq. (1.2) we are able to cancel all divergences of the action including the logarithmic one. In the AdS case the size of the asymptotic boundary always grows in equal proportion in all directions when the boundary approaches infinity, so that γ should depend only on dimension d and be the same for all possible metrics on the boundary. Note that it is kind of mystery that γ in the AdS case should be the same as in the asymptotically flat case with one S_1 component in the boundary (i.e., as in the Schwarzschild black hole case). This becomes even more surprising when we recall that for AdS space both the bulk and boundary parts in the EH action diverge while for the asymptotically flat space only the boundary part causes the divergence. The same is also true for the Lau-Mann prescription with the counterterm (1.3) where the coefficient c_{LM} (for boundary being product of a sphere and S_1 factors) should in general be $c_{LM}=-2\sqrt{\gamma/(\gamma-1)}$. Only for $\gamma=d-2$ (the boundary is $S_1\times S_{d-2}$) does there exist a correspondence between Eq. (1.3) and the AdS prescription (1.4). There must be deep reasons for the coincidence of γ's in these two cases.

III. EXAMPLES

A. Asymptotically (globally or locally) Euclidean spaces

The asymptotically (globally) Euclidean space is defined [16] to be one admitting a chart $\{x^\mu\}$ such that for $(x_\mu x^\mu)^{1/2}=\rho>\rho_0$ the metric can be written as

$$g_{\mu\nu}=\left(1+\frac{a^2}{\rho^2}\right)^2\delta_{\mu\nu}+O\left(\frac{1}{\rho^3}\right). \tag{3.1}$$

It is known that the only asymptotically globally Euclidean solution of the Einstein equations is flat space. Usually flat space-time is considered as a reference metric with respect to which one determines the contribution of a curved metric to the action. In this way, one automatically (by definition) assigns zero gravitational action to the flat space. It is then a part of the positive action theorem that in the class of asymptotically Euclidean metrics the gravitational action is zero only if the metric is flat. In our method, however, flat space stands on equal ground with any other space-times and it is not meaningless to ask what is the gravitational action for the flat space-time itself. Choose a metric on flat space R^d to take the standard form $ds^2_{R^d}=d\chi^2+\chi^2 ds^2_{S_{d-1}}$ and determine the compact space M_r^d as $0\leqslant\chi\leqslant r$. The space M_r^d has volume $V(r)=(r^d/d)\Sigma_{d-1}$ and its boundary ∂M_r^d is a round sphere S_{d-1} with area $A(r)=r^{d-1}\Sigma_{d-1}$ (we denote Σ_n to be area of an n-dimensional sphere, $\Sigma_3=2\pi^2$), so that the diameter (2.1) of M_r^d is $l^*=r/d$. We have $\gamma=d-1$ and $c(\gamma)=-2(d-1)/d$ for M_r^d. Substituting these ingredients into formula (2.10) we find that the regularized gravitational action (2.2) indeed vanishes for flat space.

In our analysis we are not restricted to consider only solutions of the Einstein equations and are interested in any metric for which the bulk integral $\int_{M_r}R$ converges for large r. An example of an asymptotically Euclidean metric with vanishing Ricci scalar R is the wormhole metric [16]

$$ds^2=\left(1+\frac{a^2}{4\rho^2}\right)^2(d\rho^2+\rho^2 ds^2_{S_3}), \tag{3.2}$$

where $ds^2_{S_3}$ is the metric of a standard three-sphere. Obviously, the condition (3.1) is satisfied for (3.2). In fact, the metric (3.2) describes space with two asymptotically Euclidean regions at $\rho\to\infty$ and $\rho\to 0$ with a minimal three-sphere located at $\rho=a/2$. One can bring the metric (3.2) to the form (2.5) by introducing the radial coordinate $\chi=\rho-a^2/4\rho$. Then Eq. (3.2) reads[2]

$$ds^2=d\chi^2+(\chi^2+a^2)ds^2_{S_3}. \tag{3.3}$$

Since the manifold has two asymptotic regions (at large negative and positive values of χ), we define the compact manifold M_r in a symmetric way as $-r\leqslant\chi\leqslant r$. The boundary ∂M_r then has two components at $\chi=-r$ and $\chi=+r$, respectively. The manifold M then is approached in the symmetric limit when $r\to\infty$. The area $A(r)$ of the boundary ∂M_r is $A(r)=2(r^2+a^2)^{3/2}\Sigma_3$. The integral of the extrinsic curvature reads $\int_{\partial M_r}K=\partial_r A(r)=6r(r^2+a^2)^{1/2}\Sigma_3$ and the EH action $W_{EH}=-(3/4\pi G)r(r^2+a^2)^{1/2}\Sigma_3$ diverges as r^2 for large r. Calculating the *diameter* l^*, Eq. (2.1), of the manifold M_r we find

$$l_r^*=\frac{r}{4}+\frac{3}{8}\frac{a^2}{r}+O\left(\frac{1}{r^3}\right). \tag{3.4}$$

In this case $\gamma=3$ and $c(\gamma)=-\frac{3}{2}$. It seems that our regularization procedure applied to the metric (3.3), according to Eq. (2.12), should lead to the action which grows linear with r. However, for the metric (3.3) the coefficients A_1 and B in the expansion (2.7) vanish. Therefore, calculating the regularized action (2.2), (2.3), (2.10) we get the finite value

[2]Being extended to higher dimension d the metric (3.3) has scalar curvature $R=(d-1)(d-4)a^2/(\chi^2+a^2)^2$ and the integral $\int_{M_r}R$ diverges as r^{d-4} for large r.

$$W_{gr}=-\frac{3\pi a^2}{4G} \tag{3.5}$$

when we take the limit of infinite r. The leading asymptotic value for Eq. (3.4) is $l_a^*=r/4$. Using this quantity in the boundary counterterm we find

$$W_{gr}^a=\frac{3\pi a^2}{4G}.$$

If outside of a compact region the metric approaches the standard flat R^d metric with boundary S_{d-1} identified under some discrete subgroup of $SO(d)$ with a free action on S_{d-1}, such metric is asymptotically locally Euclidean. Note that for both the locally and globally Euclidean metrics the parameter γ in the large r expansions (2.7), (2.8) takes its maximal possible value $\gamma=d-1$. An example of a $d=4$ asymptotically locally Euclidean solution of the Einstein equations is the Eguchi-Hanson metric [16,17]

$$ds^2=\left(1-\frac{a^4}{\rho^4}\right)^{-1}d\rho^2+\left(1-\frac{a^4}{\rho^4}\right)\times\frac{\rho^2}{4}(d\psi+\cos\theta d\phi)^2+\frac{\rho^2}{4}(d\theta^2+\sin^2\theta d\phi^2), \tag{3.6}$$

where in order to remove the apparent singularity at $\rho=a$ one should identify ψ modulo 2π rather than modulo 4π as is usual for Euler angles on S_3. This identification makes the surface of constant $\rho>a$ into projective space RP^3, i.e., a three-sphere with antipodal points identified. The surface $\rho=a$ is a two-sphere. Defying M_r as $0\le\rho\le r$ we find

$$V(r)=\frac{r^4}{32}\left(1-\frac{a^4}{r^4}\right)\Sigma_3',$$
$$A(r)=\frac{r^3}{8}\left(1-\frac{a^4}{r^4}\right)^{1/2}\Sigma_3', \tag{3.7}$$

where $\Sigma_3'=\int\sin\theta d\theta d\phi d\psi$. From Eqs. (3.7) the diameter l^* is found to be

$$l_r^*=\frac{r}{4}\left(1-\frac{a^4}{r^4}\right)^{1/2}.$$

The vector normal to ∂M_r has the components ($n^r=(1-a^4/r^4)^{1/2},0,0,0$) and we have that

$$\int_{\partial M_r}K=n^r\partial_r A(r)=\frac{3}{8}r^2-\frac{1}{8}\frac{a^4}{r^2}+O\left(\frac{1}{r^6}\right).$$

Thus, the EH action for the metric (3.6) diverges as r^2. Calculating the regularized action (2.10) (in this case $\gamma=3$) we find that the counterterm (2.3) precisely compensates the r^2 divergence while the rest of the terms vanish (as a^4/r^2) in the limit of large r. Thus, the metric (3.6) has vanishing gravitational action, $W_{gr}=0$. One obtains the same result if the asymptotic quantity l_a^* is used in the counterterm.

B. Asymptotically flat spaces

In the class of asymptotically flat metrics we include all metrics describing space-time with the boundary at infinity being an S_1 bundle over an S_{d-2}, where the S_1 fiber approaches a constant length. So the growth of the area of the boundary for large r is due to the spheric component S_{d-2} and we have $\gamma=d-2$ for all spaces of this class. For $d=4$ such bundles are labeled by the first Chern number. If it vanishes, the boundary has the topology of the direct product $S_1\times S_{d-2}$. Otherwise, its topology is more complicated. The boundary then is a squashed sphere. The fiber S_1 in the bundle is usually due to compactified Euclidean time.

1. Schwarzschild metric in d dimensions

A generalization of the four-dimensional metric (2.14) to higher dimensions is the metric

$$ds^2=g(\rho)d\tau^2+\frac{d\rho^2}{g(\rho)}+\rho^2 ds^2_{S_{d-2}},\quad g(\rho)=1-\left(\frac{\mu}{\rho}\right)^{d-3}, \tag{3.8}$$

where $0\le\tau\le2\pi\beta_H$, $\beta_H=2\mu/(d-3)$. Though the analysis can be done in terms of a metric of the type (2.5), the calculation is simpler for a metric of the form (3.8). In this coordinate system we define the compact manifold M_r as $\mu\le\rho\le r$. The area of ∂M_r and volume of M_r are

$$A(r)=2\pi\beta_H\Sigma_{d-2}r^{d-2}g^{1/2}(r),$$
$$V(r)=2\pi\beta_H\Sigma_{d-2}\frac{1}{d-1}(r^{d-1}-\mu^{d-1}). \tag{3.9}$$

For large r the area $A(r)$ grows as r^{d-2} so that[3] $\gamma=d-2$. The diameter l^* of M_r is

$$l^*(r)=\frac{r}{d-1}g^{-1/2}(r)\left[1-\left(\frac{\mu}{r}\right)^{d-1}\right]. \tag{3.10}$$

In the coordinate system (3.8) the integral of the extrinsic curvature of the boundary is given by the formula $\int_{\partial M_r}K=n^r\partial_r A(r)$, where $n^r=g^{1/2}(r)$ is the nonzero component of vector normal to ∂M_r. For finite r the regularized action (2.2) reads

$$W_{gr}=-\frac{\beta_H\Sigma_{d-2}}{8G}\left((d-3)\mu^{d-3}-\frac{2(d-2)}{r^2}\mu^{d-1}\frac{g(r)}{1-(\mu/r)^{d-1}}\right).$$

In the limit of large r it goes to the finite value

[3]Since $\gamma=d-2$, it seems that the regularized action (2.12) should diverge as $r^{\gamma-2}$. However, it happens that for the metric (3.8) the only nonzero (growing with r) terms in the expansion (2.7) for the area are $r^{\gamma-2}$ and r. Therefore, the action is indeed finite.

$$W_{gr}=-\frac{(d-3)\mu^{d-3}}{8G}\beta_H\Sigma_{d-2}. \tag{3.11}$$

The asymptotic value of the diameter (3.10) is $l_a^*=r/(d-1)$. It can be used in constructing the boundary counterterm (2.3). The corresponding regularized action

$$W_{gr}^a=-\frac{\beta_H\Sigma_{d-2}}{8G}\{(d-3)\mu^{d-3}+2(d-2)r^{d-3}[g(r)-g^{1/2}(r)]\}$$

exactly coincides with the one obtained within the standard subtraction procedure $W=-(1/8\pi G)\int_{\partial M_r}(K-K_0)$ provided the reference metric is the metric of flat space with $K_0=(d-2)/r$. For large r we obtain

$$W_{gr}^a=\frac{\mu^{d-3}}{8G}\beta_H\Sigma_{d-2}. \tag{3.12}$$

2. Taub-NUT and Taub-bolt metrics

For $d=4$ the boundary at infinity, which is the fiber bundle of S_1 over S_2, may be nontrivial if the corresponding Chern number is nonzero. It is the case for the Taub-Newman-Unti-Tamburino (Taub-NUT) and Taub-bolt metrics which can be present in the form [17]

$$ds^2=f(\rho)(d\tau+2n\cos\theta d\phi)^2+\frac{d\rho^2}{f(\rho)}+(\rho^2-n^2)ds_{S_2}^2, \tag{3.13}$$

where the metric function is

$$f(\rho)=\frac{\rho-n}{\rho+n} \tag{3.14}$$

for the Taub-NUT metric and

$$f(\rho)=\frac{(\rho-n/2)(\rho-2n)}{(\rho^2-n^2)} \tag{3.15}$$

for the Taub-bolt metric. The Euclidean time τ in Eq. (3.13) should be identified with period $8\pi n$ while the angle ϕ is identified as modulo 2π. In fact one should take two different coordinate patches which are nonsingular at the north pole ($\theta=0$) and south pole ($\theta=\pi$), respectively. The overlapping of these patches gives a surface of constant $\rho>\rho_+$ ($\rho_+=n$ for Taub-NUT and $\rho_+=2n$ for Taub-bolt), the topology of three-sphere [with $(\tau/(2n),\theta,\phi)$ being Euler angles].

The manifold M_r in the sequence of spaces approaching the space M is defined by the range $\rho_+\leqslant\rho\leqslant r$ of the radial coordinate. The square root of determinant of metric (3.13), $\sqrt{g}=(r^2-n^2)\sin\theta$, does not depend on the metric function $f(r)$. Therefore for both Eqs. (3.14) and (3.15) the volume of the space M_r is

$$V(r)=32\pi^2 n\left[\left(\frac{r^3}{3}-\frac{\rho_+^3}{3}\right)-n^2(r-\rho_+)\right]. \tag{3.16}$$

The area of the boundary ∂M_r is

$$A(r)=32\pi^2 n(r^2-n^2)f^{1/2}(r). \tag{3.17}$$

The diameter of M_r then is

$$l_{TN}^*=\frac{1}{3}(r+2n)\sqrt{\frac{r-n}{r+n}}=\frac{r}{3}+\frac{n}{3}-\frac{n^2}{2r}+O(1/r^2)$$

for Taub-NUT space and

$$l_{TB}^*=\frac{r}{3}+\frac{5}{12}n-\frac{7}{32}\frac{n^2}{r}+O(1/r^2)$$

for Taub-bolt space. In both cases the leading asymptotic value is $l_a^*=r/3$. Calculating the regularized gravitational action one obtains

$$W_{gr}^a=-W_{gr}=\frac{4\pi n^2}{G} \tag{3.18}$$

for the Taub-NUT metric and

$$W_{gr}^a=-W_{gr}=\frac{5\pi n^2}{G} \tag{3.19}$$

for the Taub-bolt metric. The expressions for W_{gr}^a agree with the results obtained in [12] within the square root prescription (1.3) and with the calculation performed in [8] using the AdS prescription. In the later case the expressions (3.18) and (3.19) are recovered in the limit of infinite AdS radius l. The difference between Eqs. (3.19) and (3.18) yields the results of [18,19]. On the other hand, Eqs. (3.18), (3.19) agree with the much older result by Gibbons and Perry [20] obtained by an ''imperfect match'' of the Taub-NUT (Newman-Unti-Tamburino) solution to Euclidean flat space.

3. Kerr-Newman metric

The Euclidean Kerr-Newman metric parametrized by mass m, electric charge q, and the rotation parameter a takes the form

$$ds^2=g_{rr}dr^2+g_{\theta\theta}d\theta^2+g_{\tau\tau}d\tau^2+2g_{t\phi}dtd\phi+g_{\phi\phi}d\phi^2,$$

$$g_{rr}=\frac{\rho^2}{\Delta},\quad g_{\theta\theta}=\rho^2,\quad g_{\tau\tau}=\rho^{-2}(\Delta+a^2\sin^2\theta),$$

$$g_{\tau\phi}=\rho^{-2}\sin^2\theta(r^2-a^2-\Delta),\quad g_{\phi\phi}=\rho^{-2}[(r^2-a^2)^2+\Delta a^2\sin^2\theta]\sin^2\theta,$$

$$\Delta(r)=r^2-a^2-q^2-2mr,\quad \rho^2=r^2-a^2\cos^2\theta. \tag{3.20}$$

This metric has vanishing scalar curvature although the Ricci tensor vanishes only if $q=0$. The Euclidean metric (3.20) can be obtained from the Lorentzian metric by taking the Wick rotation of time and supplementing this by the parameter transformation $a\to\imath a$, $q\to\imath q$. The Euclidean instanton (3.20) is a regular manifold for $r\geqslant r_+$, where $r_+=m$

$+\sqrt{m^2+a^2+q^2}$ is the positive root of the equation $\Delta(r)=0$, provided one makes certain identifications. The angle coordinate ϕ should be identified modulo 2π. We must also identify points (τ,ϕ) with $(\tau+2\pi\beta_H,\phi+2\pi\Omega\beta_H)$, where

$$\beta_H=\frac{(r_+^2-a^2)}{\sqrt{m^2+a^2+q^2}},\quad \Omega=\frac{a}{r_+^2-a^2}.$$

For the metric (3.20) one has that $\sqrt{g}=\rho^2\sin\theta$. The boundary of the manifold M_r we define by the equation $r=$const. We then find, for the volume and area,

$$V(r)=\frac{8}{3}\pi^2\beta_H[r^3-r_+^3-a^2(r-r_+)],$$

$$A(r)=4\pi^2\beta_h\Delta^{1/2}(r)I(r), \tag{3.21}$$

where $I(r)=\int_{-1}^{1}dx\sqrt{r^2-a^2x^2}$. From Eqs. (3.21) we find

$$l^*(r)=\frac{r}{3}+\frac{m}{3}+\left(\frac{m^2}{2}+\frac{q^2}{6}-\frac{a^2}{9}\right)\frac{1}{r}+O(1/r^2)$$

for the diameter of M_r. The asymptotic value is $l_a^*=r/3$.

The vector normal to the boundary ∂M_r has the component $n^r=(\Delta/\rho^2)^{1/2}$, so the extrinsic curvature of the boundary $K=\nabla_\mu n^\mu=(1/\sqrt{g})\partial_r(\sqrt{g}n^r)$. For the metric h_{ij} induced on ∂M_r we have $\sqrt{h}=\sqrt{g}(\Delta/\rho^2)^{1/2}$ and hence, $K\sqrt{h}=[r\Delta(r)/\rho^2]\sin\theta+\frac{1}{2}\Delta'\sin\theta$. Performing the integration over angle θ and taking into account that $\int d\phi d\tau=4\pi^2\beta_H$ we obtain

$$\int_{\partial M_r}K\sqrt{h}=4\pi^2\beta_H\left[\frac{\Delta(r)}{a}\ln\left(\frac{r+a}{r-a}\right)+\Delta'(r)\right].$$

Calculating now the regularized action we get

$$W_{gr}^a=-W_{gr}=\pi m\beta_H. \tag{3.22}$$

This reproduces previous results [2,18] obtained within the subtraction regularization procedure.

4. Flat space in spheroidal coordinates

In all examples considered so far the sequence of boundaries ∂M_r was chosen in a natural way for a given metric on manifold M. However, there of course exists freedom to choose different shapes for the boundaries ∂M_r. The important question arises as to how the limiting value of the regularized action $W_{gr}[M_r\rightarrow M]$ depends on the limiting sequence of the boundaries chosen. To address this question we consider flat d-dimensional space. In Sec. III A we demonstrated that choosing ∂M_r to be a round $(d-1)$-sphere of radius r the limiting value of the gravitational action is just zero. In this section we want to recalculate the action for flat space, choosing the set of boundaries ∂M_r to be now a sequence of spheroidal surfaces. We choose the spheroidal coordinates on flat space where the metric reads

$$ds^2=\frac{\rho^2}{\Delta(r)}dr^2+\rho^2d\theta^2+\Delta(r)\sin^2\theta d\phi^2+r^2\cos^2\theta ds^2_{S_{d-3}}, \tag{3.23}$$

where $\Delta(r)=r^2-a^2$, $\rho^2=r^2-a^2\cos^2\theta$. The metric (3.23) is obtained as the $\tau=$const part of the $(d+1)$-dimensional metric [21] generalizing the Kerr metric by setting the mass to zero. It is regular if $r\geqslant a$. For Eq. (3.23) we have $\sqrt{g}=\rho^2r^{d-3}\sin\theta\cos^{d-3}\theta$. We define the surface ∂M_r by the equation[4] $r=$const. Further calculations go along the same lines as for the Kerr-Newman metric. We have for the volume and area, respectively,

$$V(r)=\frac{4\pi\Sigma_{d-3}}{d(d-2)}r^d\left(1-\frac{a^2}{r^2}\right),$$

$$A(r)=4\pi\Sigma_{d-3}\frac{r^{d-1}}{(d-2)}\left[1-\left(\frac{d-1}{d}\right)\frac{a^2}{r^2}-\frac{1}{d(d+2)}\frac{a^4}{r^4}+O(a^6/r^6)\right]. \tag{3.24}$$

The large r expansion for the diameter l^* is

$$l^*(r)=\frac{r}{d}\left(1-\frac{1}{d}\frac{a^2}{r^2}-\frac{(d^2-2)}{d^2(d+2)}\frac{a^4}{r^4}+O(a^6/r^6)\right). \tag{3.25}$$

Hence, its asymptotic value is $l_a^*=r/d$ as it should be for $\gamma=d-1$. For the extrinsic curvature of ∂M_r we have $K\sqrt{h}=n^r\partial_r(\sqrt{g}n^r)$, where $n^r=(\Delta/\rho^2)^{1/2}$ is a component of the normal vector. After performing the integration over τ, ϕ, and θ we get

$$\int_{\partial M_r}K\sqrt{h}=4\pi\Sigma_{d-3}r^{d-2}\times\left(\frac{d-1}{d-2}-\frac{(d-1)}{d}\frac{a^2}{r^2}-\frac{2}{d(d+2)}\frac{a^4}{r^4}+O(a^6/r^6)\right).$$

Computing in the leading order the regularized action

$$W_{gr}^a=-\frac{\Sigma_{d-3}}{2G}\frac{(d-1)}{d(d-2)}a^2r^{d-4}[1+O(a^2/r^2)],$$

$$W_{gr}=\frac{\Sigma_{d-3}}{2G}\frac{(d+1)}{d^2(d+2)}a^4r^{d-6}[1+O(a^2/r^2)], \tag{3.26}$$

[4]The surface of constant r is a spheroidal surface with curvature depending on the angle θ. For $d=3$ we have, in particular, $\mathcal{R}=2r^2/(r^2-a^2\cos^2\theta)^2$. Note that the in the limit of infinite r the surface tends to a round sphere.

we find that the result is different for W_{gr} and W^a_{gr}. The action W_{gr} is less singular: the leading term is r^{d-6}. Let us concentrate on analysis of the regularized action W_{gr}. We observe that for $d\leqslant 5$ the limiting value of the action is zero. This is in agreement with our computation done in Sec. III A for the boundary being a round sphere. For $d=6$ the large r behavior of the action is dominated by the constant term and the limiting value is finite. For $d\geqslant 7$ the gravitational action diverges as r^{d-6}. These observations are similar to the ones made in [13]. We see that for $d\geqslant 6$ the limiting value of the action appears to depend on the choice of the limiting sequence of boundaries. A possible resolution of this problem is to find a new counterterm which may remove the leading term in Eqs. (3.26). It is not difficult to find the appropriate counterterm among terms quadratic in the boundary curvature. Indeed, the invariant $S\equiv\mathcal{R}_{ij}^2-[1/(d-1)]\mathcal{R}^2$ identically vanishes for the round $(d-1)$-dimensional sphere (in this case $\mathcal{R}_{ij}=[(d-2)/r^2]\gamma_{ij}$). Moreover, its first variation δS due to a small deviation of the metric from that of the round sphere also identically vanishes. Therefore, the geometric invariant S is nonzero only in second order with respect to the deformation of the metric of the round sphere. The spheroidal surface is an a^2-dependent deformation of the round sphere and we can expect that in the leading order (small a or large r) the invariant S is proportional to a^4. This is exactly what we need for cancellation of the divergence (3.26) also proportional to a^4. A detail computation shows that for large r we have

$$S=\frac{(d-3)^2}{(d-1)}[(d-2)\cos(\theta)^4-2\cos(\theta)^2+(d-2)]\frac{a^4}{r^8}+O\left(\frac{a^6}{r^{10}}\right).$$

Integrating this expression over angles we find that the boundary integral

$$W_{ct1}=-\frac{1}{16\pi G}\frac{d^2(d-1)}{(d-3)^2(d-2)}(l^*)^3\int_{\partial M_r}\left(\mathcal{R}_{ij}^2-\frac{1}{d-1}\mathcal{R}^2\right) \tag{3.27}$$

is that additional counterterm which can be added to the gravitational action in order to cancel the divergence (3.26). Some similarity of the counterterm (3.27) and the quadratic in the boundary curvature counterterm in the AdS prescription (1.2) should be noted.

IV. MORE OF THE BOUNDARY GEOMETRY

A. Asymptotic geometry of Ricci flat space

The universality of the AdS prescription (1.2) valid for any metric $h_{ij}(x)$ on the asymptotic boundary of AdS space is based on the possibility to well pose the Dirichlet boundary problem for the Einstein equations with positive cosmological constant. Indeed, the solution of the Einstein equations is completely determined once one fixes the induced metric on the boundary of the space (more precisely, one needs to find the manifold of negative constant curvature which has a given conformal structure at infinity). An existence theorem for such an Einstein metric was proved in [5]. One can explicitly obtain an asymptotic expansion of bulk metric near infinity starting from any metric at infinity [6]. Technically, one uses the distinguished coordinate system [4] where the bulk metric takes the form

$$ds^2=\frac{l^2d\rho^2}{4\rho^2}+\frac{1}{\rho}[h_{ij}(x)+h_{ij}^{(1)}(x)\rho+h_{ij}^{(2)}(x)\rho^2+\cdots]dx^idx^j, \tag{4.1}$$

where ρ is a radial coordinate, and $\rho=0$ determines the infinity of space. Once one picks the first term $h_{ij}(x)$ the Einstein equations determine the other terms in the ρ expansion, $h_{ij}^{(1)}(x), h_{ij}^{(2)}(x), \ldots$, as local covariant functions of the metric $h_{ij}(x)$. That is why the divergences (due to the integration in the action over small ρ) of the EH action are completely determined by the asymptotic metric $h_{ij}(x)$ and expressed as local covariant functions of $h_{ij}(x)$. The idea of introducing the counterterms determined on the regularized (staying at $\rho=\epsilon$) boundary [with boundary metric $(1/\epsilon h_{ij}(x)$] then appears quite naturally [7].

To what extent can the same be done in the asymptotically flat case? The Einstein equations then determine the metric with vanishing Ricci tensor. In analogy with the anti–de Sitter case we can fix the metric $h_{ij}(x)$ at infinity of the space and try to determine the metric in the bulk by solving the equation $R_{\mu\nu}=0$. In particular, we can use the coordinate system (ρ,x) where the metric takes a form similar to Eq. (4.1):

$$ds^2=d\rho^2+\rho^2\left(h_{ij}(x)+h_{ij}^{(1)}(x)\frac{1}{\rho}+h_{ij}^{(2)}(x)\frac{1}{\rho^2}+\cdots\right)dx^idx^j; \tag{4.2}$$

the infinity of space is at the infinite value of ρ. [Note that the boundary area for the metric (4.2) grows as ρ^{d-1} so that the metric is characterized by the value $\gamma=d-1$ of the parameter γ introduced in Sec. II.] We do not give here a detailed analysis of the problem. We just note that the bulk equations $R_{\mu\nu}=0$ determine not only the ρ evolution of the metric [i.e., how terms in the expansion (4.2) are determined by the asymptotic metric $h_{ij}(x)$] but also give a constraint on the asymptotic metric $h_{ij}(x)$. Indeed, the first term in the series (4.2) cannot be arbitrary: the bulk equations $R_{\mu\nu}=0$ dictate that it satisfy the equation

$$\mathcal{R}_{ij}[h]=(d-2)h_{ij}. \tag{4.3}$$

Thus, the infinity of Ricci flat space should have the geometry of $(d-1)$-dimensional de Sitter space which we will loosely call a round sphere. It is an important difference from the anti–de Sitter case where the boundary metric can be arbitrary from a given conformal class. We may also consider the case when the boundary has the topology of a product of n circles S_1 (the radius of each circle approaches a constant value at infinity) and $(d-n-1)$-dimensional surface Σ. The parameter $\gamma=d-n-1$ in this case. The bulk metric then is Ricci flat only if the surface Σ is $(d-n-1)$-dimensional de Sitter space.

The constraint (4.3) explains why for known examples our prescription (2.3) gives the same result as the Lau-Mann prescription (1.3). In these cases the boundary is chosen consistently with the form of the metric (4.2); i.e., it is defined as $\rho=r=$const [the boundary metric is $r^2h_{ij}(x)$]. From Eq. (4.3) we get that the integral in Eq. (1.3) is proportional to the area of the boundary in the same way as in the prescription (2.3).

B. When the boundary is not a round sphere

We saw above that the infinity of the Ricci flat space has the geometry of sphere S^{d-1}. Therefore, it is a distinguished choice to take the boundary ∂M_r to be a round sphere. This is what we were doing in most of the examples considered above. However, we are of course free to choose the boundary to be any other closed surface Σ, topologically equivalent to a sphere. An important question then is how much our regularization procedure is sensitive to this. In particular, it was claimed in Sec. II that the coefficient $c(\gamma)$ in front of the counterterm (2.3) is determined only by topology but not geometry of the boundary. However, the analysis in Sec. II was done for a boundary chosen consistently with the form of the bulk metric (2.5), i.e., defined as $\chi=$const. In the Ricci flat case, as we just have seen, it means that the boundary is a round sphere. So what happens if we take an arbitrary surface as the boundary? Should the coefficient in front of the counterterm (2.3) depend on the geometry of the surface? All these questions, in fact, challenge the universality of our prescription. A related important question (partially addressed in Sec. III) concerns the gravitational action of flat space: does it universally vanish or may it take a nonzero or even infinite value depending on the shape of the boundary?

Analyzing this problem we consider the simplest possible case when the manifold M^d is flat space with Cartesian coordinates $\{z_1,z_2,\ldots,z_d\}$. In this space we consider the sequence of surfaces Σ_r defined by equation[5]

$$\frac{z_1^2}{a_1^2}+\frac{z_2^2}{a_2^2}+\cdots+\frac{z_d^2}{a_d^2}=r^2 \tag{4.4}$$

and parametrized by radius r; when r goes to infinity the region M_r inside the surface (4.4) covers the whole manifold M. The parameters $\{a_i\}$ indicate how the surface (4.4) deviates from the round sphere S^{d-1} which is determined by Eq. (4.4) when all $a_i=1,\ i=1,\ldots,d$. It should be noted that the sequence of surfaces (4.4) is quite different from the spheroidal surfaces considered in Sec. III. The spheroidal surfaces still approach a round sphere when the radius goes to infinity while the surface (4.4) remains different from a round sphere even at infinite r.

[5]I would like to thank Rob Myers for the suggestion to consider this example and for many comments on the subject of this section.

The Einstein-Hilbert action for the region M_r^d is given by

$$W_{EH}=-\frac{1}{16\pi G}\int_{\Sigma_r}2K. \tag{4.5}$$

The trace of the extrinsic curvature K of the surface (4.4) is

$$K=\frac{\left(d-2+\sum_i \frac{1}{a_i^2}\right)}{\left(\sum_i \frac{z_i^2}{a_i^4}\right)^{1/2}}-\frac{\left(\sum_i \frac{z_i^2}{a_i^6}\right)}{\left(\sum_i \frac{z_i^2}{a_i^4}\right)^{3/2}}. \tag{4.6}$$

For the integral of Eq. (4.6) over the surface (4.4) we get the expression

$$\int_{\Sigma_r}K\sqrt{\gamma}=(d-1)r^{d-2}\Sigma_{d-1}\alpha_d(a_i), \tag{4.7}$$

where $\alpha_d(a_i)$ is some function of the parameters $\{a_i\}$. The integration in Eq. (4.7) can be performed after introducing on surface (4.4) an appropriate system of angle coordinates and reduces to elliptic type integrals. When $d=4$ and only one parameter $a_1=a$ in Eq. (4.4) differs from 1 the integration can be done explicitly and we get

$$\alpha_4(a)=\frac{2}{3}\left(a+\frac{1}{a+1}\right).$$

It seems that in order to cancel the divergence (4.7) with help of the counterterm (2.3) we need to assume that the coefficient in front of the integral in Eq. (2.3) explicitly depends on the parameters $\{a_i\}$. This would indicate that our regularization prescription (2.3) is not universal and applies (as it stands) only to a boundary which is a round sphere while in the more general case the prescription should be modified appropriately to the concrete geometry of the boundary.

It is, however, instructive to analyze the behavior of the EH action (4.5),(4.6) and the regularized action (2.2),(2.3) [with $c(\gamma)=d-2$ as it stands for a round sphere] with respect to small deviations of the parameters $\{a_i\}$ from 1. For simplicity, let us assume that only two parameters $a_1=a,a_2=b$ are different from 1. Then up to second order in $(a-1)$ and $(b-1)$ we find that

$$\begin{aligned}\alpha_d(a,b)=1+\frac{(d-2)}{d}\Big[&(a-1)+(b-1)\\&+\frac{(d-2)}{d(d+2)}((a-1)^2+(b-1)^2)\\&+\frac{(d^2-d-8)}{(d-1)}(a-1)(b-1)\Big]\\&+O((a-1),(b-1))^3\end{aligned} \tag{4.8}$$

for the function $\alpha_d(a_i)$ appearing in Eq. (4.7). On the other hand, we have

$$V=\frac{ab}{d}r^d\Sigma_{d-1} \tag{4.9}$$

for the volume of M_r^d and

$$A=r^{d-1}\Sigma_{d-1}\left[1+\frac{(d-1)}{d}[(a-1)^2+(b-1)]+\frac{(d-1)}{2d(d+2)}((a-1)^2+(b-1)^2)+2\left(\frac{d^2-5}{d-1}\right)\times(a-1)(b-1)+O((a-1),(b-1))^3\right] \tag{4.10}$$

for the area of the surface Σ_r. We are now in a position to compute the regularized gravitational action (2.2),(2.3):

$$W_{EH}+W_{ct}=-\frac{1}{16\pi G}\left(\int_\Sigma 2K-\frac{2(d-1)}{dl^*}A\right).$$

We find that

$$W_{EH}+W_{ct}=-\frac{4(d^2-1)}{d^2(d+2)}\left((a-1)^2+(b-1)^2-\frac{2}{(d-1)}(a-1)(b-1)\right)r^{d-2}\Sigma_{d-1}. \tag{4.11}$$

We see that Eq. (4.11) is still divergent but it is now quadratic in $(a-1)$ and $(b-1)$. So the counterterm (2.3) cancels the large r divergence of Eq. (4.5) in zero and first orders in $(a-1)$ and $(b-1)$. Is it possible to find a counterterm which may cancel the divergence (4.11) in the second order in $(a-1)$ and $(b-1)$? The answer is yes. The required counterterm is exactly the term (3.27) which we introduced earlier in order to cancel the divergences for the spheroidal boundary. Even the overall (dependent on dimension d) coefficient in Eq. (3.27) takes the right form. In order to prove the last statement we present here the result of the integration of the invariant $S\equiv\mathcal{R}_{ij}^2-[1/(d-1)\mathcal{R}^2$ over the surface (4.4). In the second order in $(a-1)$ and $(b-1)$ it reads

$$\int_{\Sigma_r}S=\frac{4(d-3)^2(d-2)(d+1)}{d(d+2)}r^{d-5}\Sigma_{d-1}\times\left((a-1)^2+(b-1)^2-\frac{2}{(d-1)}(a-1)(b-1)\right). \tag{4.12}$$

We see that the divergence of Eq. (4.11) is exactly canceled in second order by the counterterm (3.27) so that in the functional

$$W_{EH}+W_{ct}+W_{ct1}=r^{d-2}\Sigma_{d-1}\ \ O((a-1),(b-1))^3$$

the divergence may appear only in the third and higher orders in $(a-1)$ and $(b-1)$. In fact this is true in general when all parameters $\{a_i\}$ are different from 1. In this case the expressions (4.11) and (4.12) are proportional to the same symmetric combination $\Sigma_i(a_i-1)^2-[1/(d-1)]\Sigma_{i\neq j}(a_i-1)(a_j-1)$ and the cancellation of the divergences is evident.

Presumably, this can be continued further: we should introduce more (higher order in the boundary curvature) counterterms in order to cancel the large r divergence in next orders in (a_i-1). What we get then is an infinite series of the counterterms so that the large r divergence of the action cancels in all orders in (a_i-1). This would mean that once the gravitational action includes the whole infinite series of the counterterms it vanishes for flat space for any choice of the (topologically equivalent to sphere) boundary. We expect that the structure of this infinite series is universal and determined only by the topology of the boundary. It would be interesting to get more terms in this series and see if it converges to some compact expression of the boundary curvature.

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[1] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).
[2] S. W. Hawking, in *General Relativity*, edited by S. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[3] S. W. Hawking and G. T. Horowitz, Class. Quantum Grav. **13**, 1487 (1996).
[4] C. Fefferman and C. R. Graham, Asterisque **95**, 1 (1985).
[5] C. R. Graham and J. M. Lee, Adv. Math. **87**, 186 (1991).
[6] M. Henningson and K. Skenderis, J. High Energy Phys. **23**, 9807 (1998); Fortschr. Phys. **48**, 125 (2000).
[7] V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208**, 413 (1999).
[8] R. Emparan, C. V. Johnson, and R. C. Myers, Phys. Rev. D **60**, 104001 (1999).
[9] H. Liu and A. A. Tseytlin, Nucl. Phys. **B533**, 88 (1998).
[10] M. Banados and F. Mendes, Phys. Rev. D **58**, 104014 (1998).
[11] S. R. Lau, Phys. Rev. D **60**, 104034 (1999).
[12] R. B. Mann, Phys. Rev. D **60**, 104047 (1999).
[13] P. Kraus, F. Larsen, and R. Siebelink, Nucl. Phys. **B563**, 259 (1999).
[14] A. O. Barvinsky and S. N. Solodukhin, Nucl. Phys. **B479**, 305 (1996).
[15] L. Susskind and E. Witten, ''The holographic bound in anti-de Sitter space,'' hep-th/9805114.
[16] G. W. Gibbons and C. N. Pope, Commun. Math. Phys. **66**, 267 (1979).

[17] G. W. Gibbons and S. W. Hawking, Commun. Math. Phys. **66**, 291 (1979).
[18] C. J. Hunter, Phys. Rev. D **59**, 024009 (1998).
[19] C. J. Hunter and S. W. Hawking, Phys. Rev. D **59**, 044025 (1999).
[20] G. W. Gibbons and M. J. Perry, Phys. Rev. D **22**, 313 (1980).
[21] R. C. Myers and M. J. Perry, Ann. Phys. (N.Y.) **172**, 304 (1986).